Theory of turbomachinery

Chapter 1
Introduction: Basic Principles

*Take your choice of those that can best aid your action.* (Shakespeare, Coriolanus)
Introduction

Definition

- Turbomachinery describes machines that transfer energy between a rotor and a fluid, including both turbines and compressors (source: Wiki).

- Devices in which energy is transferred, either to, or from, a continuously flowing fluid by the dynamic action of one or more moving blade rows (Dixon)

The words rotor and continuous separate turbomachines from e.g. reciprocating (piston) engines
Introduction

Why a course on turbomachines?

- Close to all electric power is produced by turbomachines
- They consume large parts of energy used in many industrial processes
- They are integral parts of gas turbines used in e.g. aircraft engines and as (shaft-) power supply in oil and gas industry (for pumps and compressors) as well as propulsion of ships
Samples

- Windpower
- Steam turbines
- Hydropower
- Turbochargers of cars and trucks
- Vacuum cleaners
- Pumps
- Dental drills
Samples
Introduction

Classifications

- Energy may flow to the fluid (increasing velocity and/or pressure) or from the fluid producing shaft power

- Flowpath: Axial or radial machines (mixed flow)

- Changes in density, compressible or incompressible analyses.

- Impulse or reaction machines: Does the pressure change in the rotor, or in a set of nozzles before the rotor?
Samples

(a) Single stage axial flow compressor or pump
(b) Mixed flow pump
(c) Centrifugal compressor or pump
(d) Francis turbine (mixed flow type)
(e) Kaplan turbine
(f) Pelton wheel

Fig 1.1 Examples of turbobomachines
Axial-flow turbines

FIG. 4.1. Large low pressure steam turbine (Siemens)
Axial-flow turbines

FIG. 4.2. Turbine module of a modern turbofan jet engine (RR)
Axial-flow Turbines: 2-D theory

FIG. 4.3. Turbine stage velocity diagrams.

Note direction of $\alpha_2$
Coordinates

Fig 1.2: The coordinate system

\[ c_m = \sqrt{c_x^2 + c_r^2} \]

"streamwise" coordinate

(a) Meridional or side view
Coordinates (2)

(b) View along the axis

(c) View looking down onto a stream surface

Fig 1.2: The coordinate system
Velocity triangles

$U$ : Blade speed
$c$ : Absolute velocity
$w$ : Relative velocity
$\alpha$ : Absolute flow angle
$\beta$ : Relative flow angle

Fig 1.3: Velocity triangles for an axial compressor stage
Fundamental laws

- The continuity of flow equation
  - mass conservation

- First law of thermodynamics and the steady flow energy equation
  - The law of conservation of energy states that the total energy of an isolated system is constant; energy can be transformed from one form to another, but cannot be created or destroyed.

- The momentum equation, \[ F = m \cdot a \]

- The second law of thermodynamics
  - Total entropy of an isolated system always increases over time, or remains constant in ideal cases where the system is in a steady state or undergoing a reversible process
System vs. Control Volume

- **System:** A collection of matter of fixed identity
  - Always the same atoms or fluid particles
  - A specific, identifiable quantity of matter

- **Control Volume (CV):** A volume in space through which fluid may flow
  - A geometric entity independent of mass
Equation of continuity

The mass flow through a surface element $dA$:

$$dm = \rho \ c_n \ dt \ dA$$

$$c_n = c \ \cos \theta$$

So that

$$\dot{m} = \frac{dm}{dt} = \rho \ c_n \ dA$$

Or, if $A_1$ and $A_2$ are flow areas at stations 1 and 2 along a passage:

$$\dot{m} = \rho_1 \ c_{n1} A_1 = \rho_2 \ c_{n2} A_2 = \rho \ c_n A$$

Fig 1.4: Flow across an element of area
The first law

The first law of Thermodynamics: For a system that completes a cycle during which heat is supplied and work is done:

\[ \int (dQ - dW) = 0 \]

If a change is done from state 1 to state 2, energy differences must be represented by changes in internal energy:

\[ E_2 - E_1 = \int_1^2 (dQ - dW) \]

or

\[ dE = dQ - dW \]
The first law

Mass flow, $\dot{m}$, enters at 1 and exits at 2

Energy is transferred from fluid to the blades of the machine, positive work is at the rate $\dot{W}_x$

Heat transfer, $\dot{Q}$, is positive from surrounding to machine
The first law

The steady state energy equation becomes (Reynold’s transport theorem)

\[
\dot{Q} - \dot{W}_x = \dot{m} \left[ (h_2 - h_1) + \left( \frac{c_2^2 - c_1^2}{2} \right) + g(z_2 - z_1) \right]
\]

Neglecting potential energy and using total (stagnation) enthalpy:

\[
\dot{Q} - \dot{W}_x = \dot{m} (h_{02} - h_{01})
\]

For an adiabatic work producing machine (turbine): \(\dot{Q} = 0\) \(\dot{W}_x > 0\)

\[
\dot{W}_x = \dot{W}_t = \dot{m} (h_{01} - h_{02})
\]

And for adiabatic work absorbing machines (compressors): \(\dot{W}_x < 0\)

\[
\dot{W}_c = -\dot{W}_x = \dot{m} (h_{02} - h_{01})
\]
Newton's second law: The sum of all forces acting on a mass, $m$, equals the time rate of momentum change:

$$\Sigma F_x = \frac{d}{dt}(mc_x)$$

Here, only $x$-component of force and velocity is considered. For steady state the equation reduces to:

$$\Sigma F_x = \dot{m}(c_{x2} - c_{x1})$$

If shear forces (viscosity) are neglected, the Euler’s equation for one-dimensional flow can be obtained:

$$\frac{1}{\rho} \frac{dp}{dz} + c \frac{dc}{dz} + g \frac{dz}{dz} = 0$$
The momentum equation

Integrating Euler’s equation in the stream direction yields Bernoulli’s equation:

\[ \int_1^2 \frac{1}{\rho} \, dp + \frac{c_2^2 - c_1^2}{2} + g (z_2 - z_1) = 0 \]

Control volume in a streaming fluid.
Bernoulli’s equation

For an incompressible fluid (constant density) using total or stagnation pressure: \( p_0 = p + \rho \frac{c^2}{2} \)

\[
\frac{1}{\rho}(p_{01} - p_{01}) + g(z_2 - z_1) = 0
\]

Using the Head, defined as \( H = z + \frac{p_0}{\rho g} \)
reduces Bernoulli’s eq. to: \( H_2 - H_1 = 0 \)

For an compressible fluid, changes in potential are often negligible:

\[
\int_{1}^{2} \frac{1}{\rho} dp + \frac{c_2^2 - c_1^2}{2} = 0
\]

For small pressure changes (or isentropic processes) :

\( p_{02} = p_{01} = p_0 \)
Units

\[ \dot{Q} - \dot{W}_x = \dot{m} \left[ (h_2 - h_1) + \frac{(c_2^2 - c_1^2)}{2} + g(z_2 - z_1) \right] \]

\[ \dot{m}(h_2 - h_1): \frac{\text{kg}}{\text{s}} \frac{\text{J}}{\text{kg}} = \frac{\text{J}}{\text{s}} = \text{W} \]

\[ \frac{\dot{m}(c_2^2 - c_1^2)}{2}: \frac{\text{kg}}{\text{s}} \frac{\text{m}^2}{\text{s}^2} = \frac{\text{Nm}}{\text{s}} = \frac{\text{J}}{\text{s}} = \text{W} \]

\[ \dot{m}g(z_2 - z_1): \frac{\text{kg}}{\text{s}} \frac{\text{m}}{\text{s}^2} = \frac{\text{Nm}}{\text{s}} = \frac{\text{J}}{\text{s}} = \text{W} \]

1 Newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 meter per second squared.

1 Joule is the energy transferred to (or work done on) an object when a force of one newton acts on that object in the direction of its motion through a distance of one meter.
**Moment of momentum**

For a system of mass $m$, the sum of external forces acting on the system about the axis A-A is equal to the time rate of change of angular momentum:

$$\tau_A = m \frac{d}{dt} (r c_\theta)$$

Where $r$ is the distance of the mass center from the axis of rotation and $c_\theta$ is the tangential velocity component.

For one-dimensional steady flow, entering at radius $r_1$ with tangential velocity $c_{\theta_1}$ and leaving at $r_2$ with $c_{\theta_2}$:

$$\tau_A = \dot{m} \left( r_2 c_{\theta_2} - r_1 c_{\theta_1} \right)$$

Multiplication with the angular velocity $\Omega = U/r$, where $U$ is the blade speed, yields:

$$\tau_A \Omega = \dot{m} \left( U_2 c_{\theta_2} - U_1 c_{\theta_1} \right)$$
Euler’s pump and turbine equations \((1.6)\)

Control volume for a generalized turbomachine.

The work done on the fluid per unit mass (specific work) becomes:

\[
\Delta W_c = \frac{\dot{W}_c}{\dot{m}} = \frac{\tau_A \Omega}{\dot{m}} = U_2 c_{\theta 2} - U_1 c_{\theta 1} > 0
\]

\[
\Delta W_t = \frac{\dot{W}_t}{\dot{m}} = U_1 c_{\theta 1} - U_2 c_{\theta 2} > 0
\]
Example: radial pump
Velocity triangles at in- and outlet

Rotor inlet: Relative velocity tangent to blade

Rotor exit: tangential velocity induced

\[ \Delta W = \frac{\dot{W}_c}{\dot{m}} = \frac{\tau_A \Omega}{\dot{m}} = U_2 c_{\theta 2} - U_1 c_{\theta 1} = [c_{\theta 1} = 0] = U_2 c_{\theta 2} \]
Rothalpy

Combining the first law of thermodynamics and Euler’s pump equation (from Newton’s second law):

\[ \Delta W_c = \dot{W}_c/m = U_2 c_{\theta 2} - U_1 c_{\theta 1} = h_{02} - h_{01} \]

Rearranging and using the definition of stagnation enthalpy, allows the definition of the rothalpy, \( I \):

\[ h_1 + c_1^2/2 - U_1 c_{\theta 1} = h_2 + c_2^2/2 - U_2 c_{\theta 2} = I \]

Where \( I = h + c^2/2 - Uc_\theta \) does not change from entrance to exit.
The second law of Thermodynamics

*Clausius Inequality*: For a system passing through a cycle involving heat exchange,

\[ \oint \frac{dQ}{T} \leq 0 \]

where \( dQ \) is an element of heat transferred to the system at an absolute temperature \( T \).

If the entire process is reversible, \( dQ = dQ_R \), equality holds true:

\[ \oint \frac{dQ_R}{T} = 0 \]

From this, the *entropy* is defined. For a finite change of state:

\[ S_2 - S_1 = \int_{1}^{2} \frac{dQ_R}{T} \quad \text{or} \quad dS = m \, ds = \frac{dQ_R}{T} \]

\( m \) being the mass of the system.
Entropy

For steady one-dimensional flow in which the fluid goes from state 1 to state 2:

\[ \int_1^2 \frac{dQ}{T} \leq \dot{m} (s_2 - s_1) \]

For adiabatic processes, \( dQ = 0 \) and this becomes: \( s_2 \geq s_1 \)

For a system undergoing a reversible process, \( dQ = dQ_R = m \, T \, ds \) and \( dW = dW_R = m \, p \, d\nu \), the first law becomes:

\[ dE = dQ - dW = m \, T \, ds - m \, p \, d\nu \]

or with \( u = E / m \)

\[ T \, ds = du - p \, d\nu \]

Further, with \( h = u + p\nu \), \( dh = du + p \, d\nu + \nu \, dp \):

\[ T \, ds = dh - \nu \, dp \]
Definitions of efficiency

Consider a turbine: The overall efficiency can be defined as

\[ \eta_0 = \frac{\text{Mechanical energy available at coupling of output shaft in unit time}}{\text{Maximum energy difference possible for the fluid in unit time}} \]

If mechanical losses in bearings etc. are not the aim of the analyses, the isentropic or hydraulic efficiency is suitable:

\[ \eta_t = \frac{\text{Mechanical energy supplied to the rotor in unit time}}{\text{Maximum energy difference possible for the fluid in unit time}} \]

The Mechanical efficiency now becomes \( \eta_0 / \eta_t \)


**Efficiency**

From the steady flow energy equation,

\[
d\dot{Q} - d\dot{W}_x = m\left[ dh + \frac{dc^2}{2} + g \, dz \right]
\]

and the second law of thermodynamics,

\[
d\dot{Q} \leq mT \, ds = m(dh - \nu \, dp) \]

\[dQ\] can be eliminated to obtain:

\[
d\dot{W}_x \leq -m\left[ \nu \, dp + \frac{dc^2}{2} + g \, dz \right]
\]

For a turbine (positive work) this integrates to:

\[
\dot{W}_x \leq m \int_{z_1}^{z_2} \nu \, dp + \left( c_1^2 - c_2^2 \right)/2 + g(z_2 - z_1)
\]
Efficiency

Once more applying $T \, ds = dh - \nu \, dp = 0$ for the reversible adiabatic process:

$$d\dot{W}_{x,\text{max}} = -\dot{m}\left[ dh + dc^2/2 + g \, dz \right]$$

and hence the maximum work from state 1 to state 2 is:

$$\dot{W}_{x,\text{max}} = \dot{m} \int_{1}^{2} \left[ dh + dc^2/2 + g \, dz \right] = \dot{m}\left[ (h_{01} - h_{02}) + g(z_{2} - z_{1}) \right]$$

where the subscript $s$ denotes an isentropic change from state 1 to state 2.

In the incompressible case, neglecting friction losses:

$$\dot{W}_{x,\text{max}} = \dot{m}g[H_{1} - H_{2}] \quad \text{where} \quad gH = p/\rho + c^2/2 + gz$$
Efficiency

(a) Turbine expansion process

(b) Compression process

Enthalpy-entropy diagrams for turbines and compressors.
Efficiency

Neglecting potential energy terms, the actual turbine rotor specific work becomes:

\[ \Delta \dot{W}_x = \dot{W}_x / \dot{m} = h_{01} - h_{02} = h_1 - h_2 + \left( c_1^2 - c_2^2 \right) / 2 \]

And, similarly, the ideal turbine rotor specific work becomes:

\[ \Delta \dot{W}_{x,\text{max}} = \dot{W}_{x,\text{max}} / \dot{m} = h_{01} - h_{02s} = h_1 - h_{2s} + \left( c_1^2 - c_{2s}^2 \right) / 2 \]

where the subscript \( s \) denotes an isentropic change from state 1 to state 2.
Efficiency

If the kinetic energy can be made useful, we define the total-to-total efficiency as

$$\eta_{tt} = \frac{\Delta \dot{W}_x}{\Delta \dot{W}_{x,\text{max}}} = \frac{(h_{01} - h_{02})}{(h_{01} - h_{02s})}$$

Which, if the difference between inlet and outlet kinetic energies is small, reduces to

$$\eta_{tt} = \frac{(h_1 - h_2)}{(h_1 - h_{2s})}$$

If the exhaust kinetic energy is wasted, it is useful to define the total-to-static efficiency as

$$\eta_{ts} = \frac{(h_{01} - h_{02})}{(h_{01} - h_{2s})}$$

Since, here the ideal work is obtained between points 01 and 2s
Efficiency

Efficiencies of compressors are obtained from similar considerations:

\[ \eta_c = \frac{\text{Minimum work input}}{\text{Actual work input to rotor}} \]

\[ \eta_c = \frac{(h_{02s} - h_{01})}{(h_{02} - h_{01})} \]

Which, if the difference between inlet and outlet kinetic energies is small, reduces to

\[ \eta_c = \frac{(h_{2s} - h_1)}{(h_2 - h_1)} \]
Small stage or polytropic efficiency

If a *compressor* is considered to be composed of a large number of small stages, where the process goes from states 1 - x - y - .... - 2, we can define a small stage efficiency as

\[ \eta_p = \frac{\delta W_{\text{min}}}{\delta W} = \frac{(h_{xs} - h_1)}{(h_x - h_1)} = \frac{(h_{ys} - h_x)}{(h_y - h_x)} = \ldots \]

If all small stages have the same efficiency, then

\[ \eta_p = \frac{\Sigma \delta W_{\text{min}}}{\Sigma \delta W} \]

\[ \Sigma \delta W = (h_x - h_1) + (h_y - h_x) + \ldots = (h_2 - h_1) \]

and thus

\[ \eta_p = \left[ \frac{(h_{xs} - h_1) + (h_{ys} - h_x) + \ldots}{(h_2 - h_1)} \right] \]

\[ \eta_c = \frac{(h_{2s} - h_1)}{(h_2 - h_1)} \]

However, since the constant pressure curves diverge:

\[ (h_{xs} - h_1) + (h_{ys} - h_x) + \ldots > (h_{2s} - h_1) \quad \text{and} \quad \eta_p > \eta_c \]
Small stage or polytropic efficiency

If $T \, ds = dh - \nu \, dp$,
then for constant pressure:
$$(dh / ds)_p = T$$

or

At equal values of $T$:
$$(dh / ds)_p = constant$$

For a perfect gas, $h = C_p \, T$,
$$(dh / ds)_p = constant$$
for equal $h$
Expansion

Assume two identical stages with:
\( \eta = \text{const}, \Delta h_0 = \text{const} \)

Define the total turbine efficiency as:
\[
\eta_{013} = \frac{h_{01} - h_{03}}{h_{01} - h_{03,ss}} = \frac{2 \cdot \Delta h_0}{h_{01} - h_{03,ss}}
\]

By observation from the figure:
\[
(h_{02} - h_{03,ss}) > (h_{02,ss} - h_{03,ss})
\]
\[
\therefore (h_{01} - h_{03,ss}) < \frac{2 \cdot \Delta h_0}{\eta}
\]
\[
\therefore \eta_{013} > \eta
\]

Usage of polytropic and isentropic efficiencies?
Small stage efficiency for a perfect gas

Now compression!

For the isentropic process

\[ T \, ds = dh - \nu \, dp = 0 \]

and with \( h = C_p \, T \),

the polytropic efficiency becomes:

\[ \eta_p = \frac{dh_{is}}{dh} = \frac{\nu \, dp}{C_p \, dT} \]
Small stage efficiency for a perfect gas

Substituting $\nu = RT / p$ into $\eta_p = \frac{d h_{is}}{d h} = \frac{\nu \, dp}{C_p \, dT}$

$$\eta_p = \frac{R \, T \, dp}{C_p \, p \, dT}$$

And with $C_p = \gamma R / (\gamma - 1)$:

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma \eta_p} \frac{dp}{p}$$

With constant $\gamma$ and efficiency, this integrates to

$$\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma \eta_p}$$
Small stage efficiency for a perfect gas

For the ideal compression, $\eta_p = 1$, and the temperature ratio becomes:

$$\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma}$$

Which is also obtainable from $p^\gamma = \text{constant}$ and $p\nu = RT$. If this is substituted into the isentropic efficiency of compression for a perfect gas,

$$\eta_c = \frac{(T_2 - T_1)}{(T_2 - T_1)}$$

a relation between the isentropic and polytropic efficiencies is obtained:

$$\eta_c = \left[ \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} - 1 \right] / \left[ \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma \eta_p} - 1 \right]$$
Small stage efficiency for a perfect gas

**Fig. 2.8.** Relationship between isentropic (overall) efficiency, pressure ratio and small stage (polytropic) efficiency for a compressor ($\gamma = 1.4$).
Small stage efficiency for a perfect gas

For a turbine, similar analyses results in

\[ \frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{\eta_p (\gamma-1)/\gamma} \]

and

\[ \eta_t = \frac{1 - \left( \frac{p_2}{p_1} \right)^{\eta_p (\gamma-1)/\gamma}}{1 - \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma}} \]

Thus, for a turbine, the isentropic efficiency exceeds the polytropic (or small stage) efficiency.
Small stage efficiency for a perfect gas

\[ \eta_p = 0.9 \]

\[ \eta_p = 0.8 \]

\[ \eta_p = 0.7 \]

\[ \eta_p = 0.6 \]

\[ \eta_t \]

\[ \frac{p_1}{p_2} \]

Fig. 2.9. Turbine isentropic efficiency against pressure ratio for various polytropic efficiencies (\( \gamma = 1.4 \)).
Reheat factor

For e.g. steam turbines

\[ R_H = \left[ (h_1 - h_{xs}) + (h_x - h_{ys}) + \ldots \right] / (h_1 - h_{2s}) \]

i.e. the ratio of the sum of small isentropic enthalpy changes to the overall isentropic enthalpy change.

Thus:

\[ \eta_i = \frac{h_1 - h_2}{h_1 - h_{2s}} = \frac{h_1 - h_2}{\Sigma \Delta h_{is}} \cdot \frac{\Sigma \Delta h_{is}}{h_1 - h_{2s}} = \eta_p R_H \]
Reheat factor

Mollier diagram showing expansion process through a turbine split up into a number of small stages.
The inherent Unsteadiness

(a) Pressure tap at *
(b) Static pressure measurements vs time